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Structural Damage Assessment Using a Generalized Minimum Rank Perturbation Theory

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Recently, the authors proposed computationally attractive algorithms to determine the location and extent of structural damage for undamped structures assuming damage results in a localized change in stiffness properties. The algorithms make use of a finite-element model and a subset of measured eigenvalues and eigenvectors. The developed theories approach the damage location and extent problem in a decoupled fashion. First, a theory is developed to determine the location of structural damage. With location determined, a damage extent theory is then developed. The damage extent algorithm is a minimum rank perturbation, which is consistent with the effects of many classes of structural damage on a finite-element model. In this work, the concept of the minimum rank perturbation theory (MRPT) is adopted to determine the damage extent on the mass properties of an undamped structure. In addition, the MRPT is extended to the case of proportionally damped structures. For proportionally damped structures, the MRPT is used to find the damage extent in any two of the three structural property matrices (mass, damping, or stiffness). Finally, illustrative case studies using both numerical and actual experimental data are presented.

I. Introduction

THE advent of the Space Shuttle has prompted considerable attention to the design and control of large space structures. Due to the large size and complexity of envisioned structures, as well as the use of advanced materials to reduce structural weight, it may become necessary to develop a structural health monitoring system to detect and locate structural damage as it occurs. From experience gained in the machinery health monitoring field, one would expect the vibration signature of the structure, either frequency response functions and/or modal parameters, to provide useful information in determining the location and extent of structural damage.

Assume that a refined finite element model (FEM) of the structure has been developed before damage has occurred. By refined, we mean that the measured and analytical modal properties are in agreement. Next, assume that at a later date some form of structural damage has occurred. If significant, the damage will result in a change in the structures modal parameters. The question is: can the discrepancy between the original FEM modal properties and postdamage modal properties be used to ascertain structural damage?

Most prior work in damage detection has used the general framework of FEM refinement (system identification) in the development of damage assessment algorithms. The motivation behind the development of FEM refinement techniques is based on the need to validate engineering FEMs before their acceptance as the basis for final design analysis. The standard problem has been to seek a refined FEM that is as close to the original FEM and whose modal properties are in agreement with those that are measured subject to various constraints such as symmetry and sparsity preservation. A considerable amount of work in this area has been published. Extensive literature surveys can be found in Refs. 1 and 2.

Recently, the authors proposed a computationally attractive technique that approaches the damage location and extent problem in a decoupled fashion.[2,3] In this work, a theory is first developed to solve the damage location problem based on the original n degree of freedom finite-element model and p measured modes, $p << n$. Next, a computationally attractive algorithm which makes use of the knowledge of location is formulated to determine the extent of damage. A unique solution for the case of incomplete measurements is arrived at by enforcing a minimum rank perturbation constraint. This minimum rank perturbation constraint is consistent with the effects of many forms of structural damage on a FEM.

In Refs. 2 and 3, the use of the minimum rank perturbation theory (MRPT) is limited to predicting the damage extent of the stiffness properties of undamped structures. In the work presented in this paper, the concept of the MRPT is extended to determine the damage extent on the mass properties of undamped structures. The application of the MRPT is further extended to determine damage extent in structures that exhibit proportional damping. For proportionally damped structures, the MRPT is used to find the perturbation due to damage in any two of the three structural property matrices (mass, damping, or stiffness).

II. Damage Detection: Location

A detailed development of the damage location algorithm is discussed in Refs. 2 and 3. For completeness, a brief overview of the algorithm is provided. In the formulation of the damage location algorithm, the structure under consideration is assumed to have an n-DOF damped nongyroscopic and noncirculatory FEM. It is also assumed that this FEM has been refined to enforce the correlation of its modal properties with those measured experimentally. Then, as shown in Refs. 2 and 3, the eigenvalue problem of a damaged structure for the ith mode can be rearranged into the form

$$d_i \equiv Z_{d_i} v_{d_i} \tag{1a}$$

$$= (\lambda_{d_i}^2 \Delta M_d + \lambda_{d_i} \Delta D_d + \Delta K_d) v_{d_i} \tag{1b}$$

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where $Z_{d_i} \equiv \lambda_{d_i}^2 M + \lambda_{d_i} D + K$. The matrices M, D, and K are given $n \times n$ real symmetric refined analytical mass, damping, and stiffness matrices of the predamaged healthy structure. The $n \times n$ real symmetric matrices ΔM_d, ΔD_d, and ΔK_d are the perturbations to the mass, damping, and stiffness matrices, respectively, which reflect the nature of the structural damage. The variables $\boldsymbol{v}_{d_i}$ and λ_{d_i} are the ith damaged eigenvector and eigenvalue measured experimentally from a postdamage modal survey. Note that the vector $\boldsymbol{d}_i$ can be determined using known quantities by using Eq. (1a).

In this paper, it is assumed that the dimension of the measured eigenvectors is the same as the analytical eigenvectors. This is true when all FEM DOF are measured, after the application of an eigenvector expansion algorithm,[1,4–6] or after the application of a FEM reduction algorithm.[7,8] The ideal situation would be to measure all FEM DOFs since the eigenvector expansion process would introduce additional errors in the expanded eigenvectors and the model reduction process would introduce errors in the FEM. It should be noted that in both cases the additional errors may become significant as the ratio of measured to unmeasured DOFs become smaller.

When the measured eigendata are not corrupted by noise, an inspection of vector $\boldsymbol{d}_i$ in terms of Eq. (1b) reveals that the jth element of $\boldsymbol{d}_i$ will be zero when the jth rows of the perturbation matrices are zero, i.e., the jth degree of freedom is not directly affected by damage. Conversely, a degree of freedom whose FEM has been affected by damage will result in a nonzero entry in $\boldsymbol{d}_i$. This damage location algorithm is essentially the modal force error criteria as proposed by Ojalvo and Pilon.[9]

In practice, the perfect zero/nonzero pattern of the damage vector $\boldsymbol{d}_i$ rarely occurs due to measurement errors in the damaged eigenvalues and eigenvectors. To provide an alternative view of the state of damage, Eq. (1a) is rewritten as

$$d_i^j \equiv z_{d_i}^j \boldsymbol{v}_{d_i} = \|z_{d_i}^j\| \|\boldsymbol{v}_{d_i}\| \cos(\theta_i^j) \tag{2}$$

where d_i^j is the jth component (or jth DOF) of the ith damage vector, $z_{d_i}^j$ is the jth row of the matrix Z_{d_i} and θ_i^j is the angle between the vectors $z_{d_i}^j$ and $\boldsymbol{v}_{d_i}$. In the case when the measurements are free of error, a zero d_i^j corresponds to a θ_i^j of ninety degrees, whereas a nonzero d_i^j corresponds to a θ_i^j different from ninety degrees. Errors in the experimental measurements of modal parameters will cause slight perturbations in the angles θ_i^j that destroy the zero/nonzero pattern of the damage vector. One would expect that the components of $\boldsymbol{d}_i$ corresponding to the damaged DOFs would be substantially larger than the other elements. However, by inspecting Eq. (2), a larger $\boldsymbol{d}_i$ component could be due to a $z_{d_i}^j$ row norm substantially larger than other rows of Z_{d_i}. Hence, when dealing with a structure whose FEM results in $z_{d_i}^j$ row norms which are of different orders of magnitude, it is more reasonable to use the deviation of the angle, θ_i^j, from ninety degrees to locate the damage.

When the number of measured modes, p, is greater than one, two different composite damage vectors may be defined as

$$\boldsymbol{d} = \frac{1}{p} \sum_{i=1}^{p} \frac{|\boldsymbol{d}_i|}{\|\boldsymbol{v}_{d_i}\|} \tag{3a}$$

$$\boldsymbol{\alpha} = \frac{1}{p} \sum_{i=1}^{p} |\boldsymbol{\alpha}_i| \tag{3b}$$

where $\alpha_i^j = \theta_i^j - 90$ deg.

It should be noted that in the multimode measurement case, Eq. (3b) is preferable when the values of $\|z_{d_i}^j\|$ have different orders of magnitude for each measured mode.

III. Damage Detection: Extent

It is sometimes necessary to determine the extent of structural damage. Structural damage often occurs at discrete locations. The effect of damage on the analytical model is often restricted to just a few elements of the finite element model. The rank of each element mass, damping, or stiffness matrix is dependent on the number of degrees of freedom defined by the element and the shape functions utilized. However, it should be noted that in general the element matrices are not of full rank. For example, the rank of the 6×6 element stiffness matrix of a three-dimensional truss element is just one. Thus, instead of using the matrix Frobenius norm minimization formulation to arrive at unique perturbation matrices, minimum rank perturbation constraints are enforced.

Assume that p damaged eigenvalues and eigenvectors have been measured. Equations (1a) and (1b) can be written in matrix form, for all p measured modes, as

$$\begin{gathered} MV_d\Lambda_d^2 + DV_d\Lambda_d + KV_d = \Delta M_d V_d \Lambda_d^2 + \Delta D_d V_d \Lambda_d \\ + \Delta K_d V_d \equiv B \\ \Lambda_d = \operatorname{diag}(\lambda_{d1}, \lambda_{d2}, \ldots, \lambda_{dp}) \\ V_d = [\boldsymbol{v}_{d1}, \boldsymbol{v}_{d2}, \ldots, \boldsymbol{v}_{dp}] \\ B = [\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_p] \end{gathered} \tag{4}$$

Note that matrix B can be determined from the original analytical model (M, D, K) and the p measured eigenvalues and eigenvectors.

A. Theoretical Background

In this section, the theoretical foundation of the MRPT is derived. This theory will be extensively used throughout the remainder of this paper.

Proposition 1

Suppose that $X, Y, \in \mathbb{R}^{n \times p}$ are given where $p < n$ and $\operatorname{rank}(X) = \operatorname{rank}(Y) = p$. Define $\mathcal{H}$ to be the set of matrices A in $\mathbb{R}^{n \times n}$ that satisfy

$$AX = Y \qquad \text{with} \qquad A^T = A \tag{5}$$

Then,

1a) If the set $\mathcal{H}$ is nonempty, the minimum rank of any matrix, A, in $\mathcal{H}$ is p. Next, define $\mathcal{H}^p$ to be a subset of $\mathcal{H}$ comprised of all A such that $\operatorname{rank}(A) = p$.
Then

1b) If the matrix Y^TX is symmetric, then one member of $\mathcal{H}^p$ is given by

$$A^p = YHY^T \qquad \text{with} \qquad H = (Y^TX)^{-1} \tag{6}$$

and

1c) The matrix defined by Eq. (6) is the unique member of $\mathcal{H}^p$.

Proof

To prove Proposition 1a, note that Eq. (5) is exactly satisfied if and only if range(Y) is included in range(A), which is also range(A^T) by symmetry. This implies that rank(Y) $= p \leq$ rank(A).

To investigate Proposition 1b, assume that the expanded singular value decomposition of one member, $A^{p,j}$, of $\mathcal{H}^p$ to be of the form

$$\begin{gathered} A^{p,j} = U^j \Sigma^j U^{j^T} \\ U^j = [\boldsymbol{u}_1^j, \boldsymbol{u}_2^j, \ldots, \boldsymbol{u}_p^j] \\ \Sigma^j = \operatorname{diag}(\sigma_1^j, \sigma_2^j, \ldots, \sigma_p^j) \end{gathered} \tag{7}$$

where the superscript j indicates the jth family member of $\mathcal{H}^p$, the $\boldsymbol{u}_i^j$ are the left and right singular vectors and the σ_i^j are the nonzero singular values of $A^{p,j}$. In the expanded singular value decomposition, the $(p + 1)$ to n singular vectors are not shown in the factorization because of their corresponding zero singular values. Note that the left and right singular vectors are the same because $A^{p,j}$ is restricted to be symmetric. For Eq. (5) to be satisfied, the range of Y, $A^{p,j}$, A^{p,j^T} and U^j must be equal. Thus, any column of Y can be written as a linear combination of the u_i^j's. The matrices Y and U^j are then related by a unique $p \times p$ invertible matrix Q^j

$$Y = U^j Q^j \tag{8}$$

Substituting Eq. (8) into Eq. (7) gives

$$A^{p,j} = Y(Q^{j-1}\Sigma^j Q^{j-T})Y^T = YH^jY^T \tag{9}$$

Thus, each family member is uniquely defined by the factorization of Eq. (6). From Eq. (9), it is evident that H^j is of full rank because its inverse exists ($H^{j-1} = Q^{jT}\Sigma^{j-1}Q^j$).

Inspection of Eq. (9) reveals that the only unknown term in the factorization is H^j. By using the factorization of $A^{p,j}$ as defined by Eq. (9), Eq. (5) can be rewritten as

$$Y = A^{p,j}X = (YH^jY^T)X = Y(H^jY^TX) \tag{10}$$

Equation (10) is satisfied if and only if $H^jY^TX = I_{p \times p}$, where $I_{p \times p}$ is the $p \times p$ identity matrix. This is true because Y and X are of full column rank. Thus, H^j is uniquely calculated to be

$$H^j = (Y^TX)^{-1} \tag{11}$$

Proposition 1c follows immediately by inspecting the right-hand side of Eq. (11). Inspection reveals that H^j is the same for all members of $\mathcal{H}^p$. This fact, in conjunction with Eq. (9), leads to the conclusion that $A^{p,j}$ is the unique member of the set $\mathcal{H}^p$. This member is given by Eq. (6).

B. Undamped Structures

In some cases, the damping of the system under consideration is assumed to be negligible. For this type of system, MRPT-based algorithms are developed assuming that the structural damage affects only the mass properties or only the stiffness properties.

1. *Damage Extent: Mass Properties*

In this case, it is assumed that the effect of damage on the stiffness properties of the structure is negligible. With this assumption, Eq. (4) can be rewritten as

$$MV_d\Lambda_d^2 + KV_d = \Delta M_d V_d \Lambda_d^2 \equiv B \tag{12}$$

Note that the eigenvectors are real and the eigenvalues are pure imaginary. Further, the eigenvectors are linearly independent, which implies that the matrix product $V_d\Lambda_d^2$ is of full column rank if rigid body modes are not included. Assume, for the moment, that B is of full rank (rank$(B) = p$). Then, Proposition 1 can be applied to determine the perturbation matrix, ΔM_d, as

$$\Delta M_d = B(B^TV_d\Lambda_d^2)^{-1}B^T \tag{13}$$

by letting $Y = B$ and $X = V_d\Lambda_d^2$. Note that the required inversion is that of a $p \times p$ matrix, where p is the number of measured modes.

Proposition 1c in conjunction with Proposition 1b implies that the existence of the unique symmetric rank p solution requires the symmetry of the matrix product $B^TV_d\Lambda_d^2$. This matrix product will be symmetric if the eigenvectors are stiffness orthogonal, i.e., the eigenvectors are orthogonal with respect to the original stiffness matrix. To see this, consider the symmetric equivalence

$$B^TV_d\Lambda_d^2 \equiv \Lambda_d^2V_d^TB \tag{14}$$

Substituting the expression for B from Eq. (12) into Eq. (14) gives

$$\Lambda_d^2V_d^TMV_d\Lambda_d^2 + V_d^TKV_d\Lambda_d^2 \equiv \Lambda_d^2V_d^TMV_d\Lambda_d^2 + \Lambda_d^2V_d^TKV_d \tag{15}$$

where the symmetry of M, K, and Λ_d^2 has been used in writing Eq. (15). From Eq. (15), it is clear that the equivalence is true if

$$(V_d^TKV_d)\Lambda_d^2 \equiv \Lambda_d^2(V_d^TKV_d) \tag{16}$$

Equation (16) will obviously be satisfied if the measured eigenvectors are stiffness orthogonal. Baruch and Bar Itzhack[10] treated one approach to mass orthogonalize the measured eigenvectors. This approach is entirely compatible with orthogonalizing the measured eigenvectors with respect to the stiffness matrix.

At this point, the extent algorithm defined previously assumes that the matrix B is of full column rank. The next proposition addresses the case in which the experimental measurements in conjunction with the matrix B as defined in Eq. (12) does not produce a B of full column rank.

Proposition 2

Suppose that $V_d, B \in \mathbb{R}^{n \times m}$, are given matrices with rank$(V_d) = m$ and rank$(B) = p$, where $p < m < n$. Further, suppose $\Lambda_d^2 \in \mathbb{R}^{m \times m}$ is a diagonal matrix of rank m. Define $\mathcal{U}$ to be the set of all matrices, ΔM_d^p, that satisfies the problem

$$\Delta M_d^p V_d^p \Lambda_d^{p^2} = B^p \quad \text{with} \quad \Delta M_d^{p^T} = \Delta M_d^p \tag{17}$$

where the superscript p indicates a rank p matrix. It is assumed that V_d is stiffness orthogonal such that the set $\mathcal{U}$ is nonempty. In Eq. (17), V_d^p and $B^p \in \mathbb{R}^{n \times p}$ and $\Lambda_d^{p^2} \in \mathbb{R}^{p \times p}$ are corresponding full rank submatrices of V_d, B, and Λ_d^2. Then the set $\mathcal{U}$ contains a single member, ΔM_d^p, that can be calculated from Eq. (13) using any V_d^p and B^p and $\Lambda_d^{p^2}$.

Proof

The jth member of the set $\mathcal{U}$ is given by

$$\Delta M_d^{p,j} = B^{p,j}H^jB^{p,j^T} \quad \text{with} \quad H^j = (B^{p,j^T}V_d^{p,j}\Lambda_d^{p,j^2})^{-1} \tag{18}$$

where the additional superscript $(\ )^{-j}$ indicates the jth member of $\mathcal{U}$. The range of any B^p is equal to the range of B, thus the $B^{p,i}$ and $B^{p,j}$ are related by

$$B^{p,i} = B^{p,j}Q^{i,j} \tag{19}$$

where $Q^{i,j} \in \mathbb{R}^{p \times p}$ and rank$(Q^{i,j}) = p$. By utilizing Eq. (19), Eq. (18) can be written for the ith member of $\mathcal{U}$ as

$$\Delta M_d^{p,i} = B^{p,j}(Q^{i,j}H^iQ^{i,j^T})B^{p,j^T}$$
$$\text{with} \quad H^i = (Q^{i,j^T}B^{p,j^T}V_d^{p,i}\Lambda_d^{p,i^2})^{-1} \tag{20}$$

In comparing Eqs. (18) and (20), it is seen that $\Delta M_d^{p,i} = \Delta M_d^{p,j}$ if

$$H^j = Q^{i,j}H^iQ^{i,j^T} \tag{21}$$

or equivalently,

$$H^{j-1} = Q^{i,j^{-T}}H^{i-T}Q^{i,j^{-1}} \tag{22}$$

where Eq. (22) makes use of the symmetry of H^i. By using the definition of B from Eq. (12), $B^{p,j}$ can be written explicitly as

$$B^{p,j} = MV_d^{p,j}\Lambda_d^{p,j^2} + KV_d^{p,j} = (MV_d^{p,i}\Lambda_d^{p,i^2} + KV_d^{p,i})Q^{i,j^{-1}} \tag{23}$$

where the latter expression of Eq. (23) utilizes Eq. (19). By using the definitions H^i and H^j, Eq. (22) can be rewritten as

$$Q^{i,j^{-T}}(\Lambda_d^{p,i^2} V_d^{p,i^T} M V_d^{p,j} \Lambda_d^{p,j^2} + \Lambda_d^{p,i^2} V_d^{p,i^T} K V_d^{p,j}) = Q^{i,j^{-T}}(\Lambda_d^{p,i^2} V_d^{p,i^T} M V_d^{p,j} \Lambda_d^{p,j^2} + V_d^{p,j^T} K V_d^{p,i} \Lambda_d^{p,j^2}) \tag{24}$$

Recognizing the first term in each parenthesis to be the same, the only condition such that $\Delta M_d^{p,j} = \Delta M_d^{p,i}$ is

$$\Lambda_d^{p,i^2} V_d^{p,i^T} K V_d^{p,j} = V_d^{p,j^T} K V_d^{p,i} \Lambda_d^{p,j^2} \tag{25}$$

There are three distinct cases for which Eq. (25) must be investigated. The first is the trivial case, in which $V_d^{p,j} = V_d^{p,i}$ (same eigenvectors used), and thus $\Lambda_d^{p,j^2} = \Lambda_d^{p,i^2}$. Under these conditions, it is clear that Eq. (25) is satisfied. The next case to be examined is that $V_d^{p,j}$ and $V_d^{p,i}$ have no common eigenvectors. This possibility could occur only if $m \geq 2p$. In this situation, each side of Eq. (25) is identically the zero matrix because the eigenvectors are assumed to be stiffness orthogonal. The final case is when $V_d^{p,j}$ and $V_d^{p,i}$ have a common subset of eigenvectors. Due to the stiffness orthogonality condition for the uncommon subset of $V_d^{p,j}$ and $V_d^{p,i}$, as well as the fact that Λ_d^{p,j^2} and Λ_d^{p,i^2} have a common subset of eigenvalues, it is trivial to show by partitioning common and uncommon eigenvectors/eigenvalues that Eq. (25) is satisfied. Equation (25) holds even in the general case that the common eigenvectors are placed in $V_d^{p,j}$ and $V_d^{p,i}$ in different column locations. Thus, not only is the calculated perturbation matrix independent of which p eigenvectors (and corresponding columns of B) are used, but also independent of the column order used to construct V_d^p.

Structures often exhibit rigid-body modes of vibration. The refined finite element model (FEM) defined by the original mass and stiffness matrix along with the perturbation mass matrix computed using Eq. (13) preserve the rigid-body characteristics. This is apparent in that the original stiffness matrix is unchanged and that the rigid-body modes are defined as modes whose eigenvectors lie in the null space of the stiffness matrix.

2. *Damage Extent: Stiffness Properties*

Here, it is assumed that the effect of damage on the mass properties of the structure is negligible. With the assumption that the structure is undamped, Eq. (4) can be rewritten as,

$$MV_d\Lambda_d^2 + KV_d = \Delta K_d V_d \equiv B \tag{26}$$

This problem is fully treated in Refs. 2 and 3. For the sake of completeness and convenience, the results are reported here. By applying the theory in Proposition 1, the perturbation to the original stiffness matrix is found to be,

$$\Delta K_d = B(B^T V_d)^{-1} B^T \tag{27}$$

In Eq. (27), it is assumed that B is of full column rank.

The properties associated with ΔK_d are as follows:

1. The matrix ΔK_d will be symmetric if the eigenvectors are mass orthogonal.
2. When the matrix B, defined earlier, is not of full rank, corresponding submatrices of B and V_d which have the same rank as B can be used in Eq. (27). Furthermore, the computed ΔK_d matrix is independent of the submatrices used if the eigenvectors are mass orthogonal.
3. The refined FEM defined by the original mass and stiffness matrices and the perturbation stiffness matrix, ΔK_d, preserves the rigid-body mode characteristics if the measured eigenvectors and the rigid-body modes are mass orthogonal.

The proofs of the above three properties are treated in Refs. 2 and 3. Note that the proofs of Properties 1) and 2) are similar to the one presented for ΔM_d in Sec. III.B.1.

C. Proportionally Damped Structures

Since many structures have nonnegligible damping, it is of practical interest to extend the application of the MRPT to address damped structures. In this analysis, the structure under consideration is assumed to exhibit proportional damping. Further, it is assumed that the structural damage has only affected two of the three property matrices of the structure.

1. *Damage Extent: Stiffness and Damping Properties*

It is assumed that the effect of the structural damage on the mass properties is negligible. With this assumption, Eq. (4) is rewritten as

$$MV_d\Lambda_d^2 + DV_d\Lambda_d + KV_d = \Delta D_d V_d \Lambda_d + \Delta K_d V_d \equiv B \tag{28}$$

The complex conjugate of Eq. (28) is

$$\Delta D_d V_d \bar{\Lambda}_d + \Delta K_d V_d = \bar{B} \tag{29}$$

where the operator $(\bar{\ })$ indicates the complex conjugate operator and the fact that ΔD_d, ΔK_d, and V_d are real has been used in writing Eq. (29). Subtracting Eq. (29) from Eq. (28) gives

$$\Delta D_d V_d(\Lambda_d - \bar{\Lambda}_d) = (B - \bar{B}) \tag{30}$$

If $(B - \bar{B})$ is assumed to be of full rank, Proposition 1 can be applied to determine the perturbation matrix, ΔD_d, as

$$\Delta D_d = (B - \bar{B}) H_d (B - \bar{B})^T$$
$$\text{with} \quad H_d = [(B - \bar{B})^T V_d(\Lambda_d - \bar{\Lambda}_d)]^{-1} \tag{31}$$

Note that ΔD_d as defined by Eq. (31) is real. Postmultiplying Eq. (28) by Λ_d and Eq. (29) by Λ_d, and subtracting the two equations leads to

$$\Delta K_d V_d(\bar{\Lambda}_d - \Lambda_d) = (B\bar{\Lambda}_d - \bar{B}\Lambda_d) \tag{32}$$

where the fact that $\bar{\Lambda}_d$ and Λ_d are diagonal matrices has been used in writing Eq. (32). If $(B\Lambda_d - \bar{B}\Lambda_d)$ is assumed to be of full rank, Proposition 1 can also be applied to determine the perturbation matrix, ΔK_d, as

$$\Delta K_d = (B\bar{\Lambda}_d - \bar{B}\Lambda_d) H_k (B\bar{\Lambda}_d - \bar{B}\Lambda_d)^T$$
$$\text{with} \quad H_k = [(B\bar{\Lambda}_d - \bar{B}\Lambda_d)^T V_d(\bar{\Lambda}_d - \Lambda_d)]^{-1} \tag{33}$$

Note that ΔK_d as defined by Eq. (33) is also real.

2. *Damage Extent: Mass and Damping Properties*

In this case it is assumed that the effect of the structural damage on the stiffness properties is negligible. With this assumption, Eq. (4) is rewritten as

$$MV_d\Lambda_d^2 + DV_d\Lambda_d + KV_d = \Delta M_d V_d \Lambda_d^2 + \Delta D_d V_d \Lambda_d \equiv B \tag{34}$$

In a similar fashion to that in the preceding section, ΔM_d and ΔD_d are determined to be

$$\Delta M_d = (B\bar{\Lambda}_d - \bar{B}\Lambda_d) H_m (B\bar{\Lambda}_d - \bar{B}\Lambda_d)^T$$
$$\text{with} \quad H_m = [(B\bar{\Lambda}_d - \bar{B}\Lambda_d)^T V_d(\Lambda_d^2\bar{\Lambda}_d - \bar{\Lambda}_d^2\Lambda_d)]^{-1} \tag{35}$$

$$\Delta D_d = (B\bar{\Lambda}_d^2 - \bar{B}\Lambda_d^2) H_d (B\bar{\Lambda}_d^2 - \bar{B}\Lambda_d^2)^T$$
$$\text{with} \quad H_d = [(B\bar{\Lambda}_d^2 - \bar{B}\Lambda_d^2)^T V_d(\Lambda_d\bar{\Lambda}_d^2 - \bar{\Lambda}_d\Lambda_d^2)]^{-1} \tag{36}$$

Note that ΔM_d and ΔD_d as defined by Eqs. (35) and (36) are real.

3. Damage Extent: Mass and Stiffness Properties

In this problem, it is assumed that the effect of the structural damage on the damping properties is negligible. With this assumption, Eq. (4) is rewritten as

$$MV_d\Lambda_d^2 + DV_d\Lambda_d + KV_d = \Delta M_d V_d \Lambda_d^2 + \Delta K_d V_d \equiv B \quad (37)$$

In a similar fashion as in Sec. III.C.1., ΔM_d and ΔK_d can be computed as follows:

$$\Delta M_d = (B - \bar{B})H_m(B - \bar{B})^T$$
$$\text{with} \quad H_m = [(B - \bar{B})^T V_d(\Lambda_d^2 - \bar{\Lambda}_d^2)]^{-1} \quad (38)$$

$$\Delta K_d = (B\bar{\Lambda}_d^2 - \bar{B}\Lambda_d^2)H_k(B\bar{\Lambda}_d^2 - \bar{B}\Lambda_d^2)^T$$
$$\text{with} \quad H_k = [(B\bar{\Lambda}_d^2 - \bar{B}\Lambda_d^2)^T V_d(\bar{\Lambda}_d^2 - \Lambda_d^2)]^{-1} \quad (39)$$

Note that ΔM_d and ΔK_d as defined by Eqs. (38) and (39) are real.

Note that the properties associated with the preceding three variations of the proposed method are very similar. For any of the three variations, these properties are as follows:

1) The perturbation matrices will be symmetric if the eigenvectors are orthogonal with respect to the unperturbed property matrix (matrix unaffected by damage).

2) The refined FEM defined by the original mass, damping, and stiffness matrix, and the computed perturbation matrices will preserve the rigid-body modes if all eigenvectors are orthogonal with respect to the unperturbed property matrix.

3) Let the equation $AX = Y$ be any of the equations from which the perturbation matrices are calculated. When the matrix Y is not of full rank, corresponding submatrices of Y and X which have the same rank as Y can be used in computing A. Further, the computed A matrix is independent of the submatrices used if the eigenvectors are orthogonal with respect to the unperturbed property matrix.

The detailed proofs of the preceding three properties can be found in Ref. 11. These proofs are not reported here, however they follow very much the same pattern as the proofs in Sec. III.B.1.

D. Computational Issues

A careful review of the proposed damage location and extent algorithms indicates the computational efficiency of the approach. The damage location algorithm requires only simple matrix operations such as matrix multiplication and addition. In addition to simple matrix operations, the damage extent algorithm requires the inversion of a $p \times p$ matrix, where p is the number of measured modes used in the extent calculation. Even for complex structures, the number of measured modes (p) is typically small.

However, it must be noted that this assessment of computational efficiency assumes that the finite element model already exists, that a modal test/identification of the structure has been performed, and that the mismatch in the number of measured DOFs vs FEM modeled DOFs has been corrected using either eigenvector expansion and/or FEM model reduction. The time/computational burden of these preprocessing steps is obviously dependent on the complexity of the actual structure under consideration.

IV. Examples

Two example problems are presented to illustrate the characteristics of some of the developed theories. The first problem is a computer simulated example. In this example, a proportionally damped model of a 50-bay, two-dimensional truss is used to illustrate the problem of damage detection in the damping and stiffness properties. The second problem involves the determination of a discrete mass loss in a laboratory mass loaded cantilevered beam using actual experimental data.

A. Simulated Example: Two-Dimensional Truss

The 50-bay, two-dimensional truss used in this example is shown in Fig. 1. The geometric and material properties of the truss are given in the figure. Each truss member was modeled as a rod element. The finite-element model of the structure has 201 degrees of freedom. The damping of the healthy model is proportional and equal to 1×10^{-1} times the healthy mass matrix plus 5×10^{-7} times the healthy stiffness matrix. In this example, damage is simulated by reducing the Young's modulus of two members. One of the damaged members is the upper longeron of the third bay. In this member, the Young's modulus is decreased from $E = 29 \times 10^6$ psi to 1×10^3 psi. The other member is the lower longeron of bay forty. This member is subjected to a complete loss of stiffness (Young's modulus equal to zero). The damping of the damaged model is also proportional and equal to 1×10^{-1} times the healthy mass matrix plus 5×10^{-7} times the damaged stiffness matrix. Thus, the damage simulated is only affecting the stiffness and damping properties of the structure. Note that the damage in the damping properties is proportional to the damage in the stiffness properties.

For our damage assessment analysis, it is assumed that only the first ten modes of vibration are measured. This problem is investigated for three different scenarios. Each scenario corresponds to a different level of random noise added to the damaged eigenvectors. In practice the noise in the eigenvectors could be due to both measurement and/or expansion errors. The first task in this analysis is to use the damage location algorithm to determine the location of the damage. Cumulative damage location vectors are first computed for all three scenarios using the modal properties of the ten damaged modes. The upper-left plot of Fig. 2 represents the exact damage location computed from the exact damage perturbation matrices (ΔK_d and ΔD_d). The upper-right plot corresponds to the case where the exact damaged eigenvector information is provided to the damage location algorithm. The lower-left and -right plots correspond to the cases where the exact damaged eigenvectors have been corrupted with 2.5% and 5% random noise, respectively. In practice the noise in

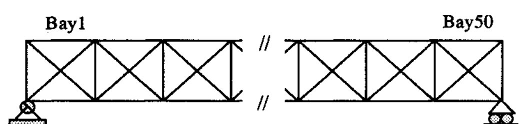

Fig. 1 Fifty-bay truss.

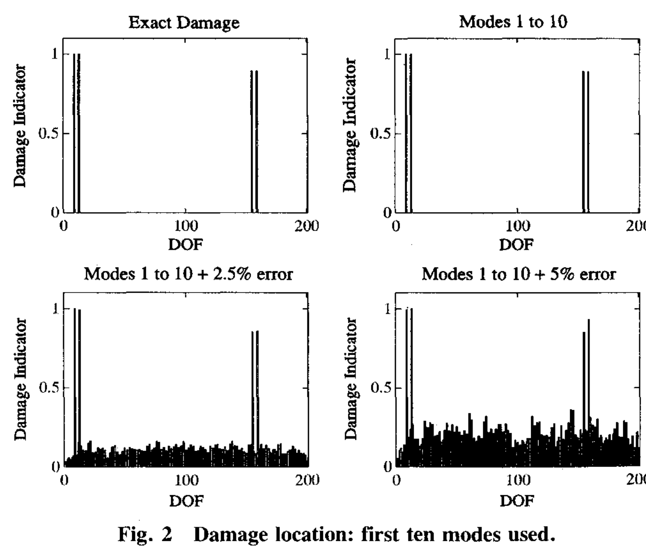

Fig. 2 Damage location: first ten modes used.

eigenvector information could be due to both measurement and/or expansion errors. As shown in Fig. 2, the location algorithm is able to exactly locate the damage when presented with noise-free information. Although not as clean, the damage can still be clearly located in the noisy eigenvector cases.

With knowledge of the location of damage, the rank of the true perturbation matrix, ΔK_d (or ΔD_d), can be found by adding the rank of the element stiffness (or damping) matrix of the damaged members. Hence, the rank of the perturbation to the stiffness matrix due to damage is two because two members having rank one element matrices are damaged. Because the damping of the structure is proportional and the mass properties are unaffected by the damage, it is deduced that the rank of the perturbation to the damping matrix is the same as the rank of the perturbation to the stiffness matrix. From Propositions 1 and 2, it is clear that only experimental data from two modes of vibration are needed to compute the extent of the damage. In the noisy situations, the two modes that should be used are the ones that most cleanly demonstrate the damage shown in Fig. 2. These modes can be determined by inspecting the individual damage vectors. An inspection of the individual damage vectors associated with the noisy eigenvectors suggests that determined that modes 8 and 9 provide the best insight into the state of the damage. The results of applying the extent algorithm [Eq. (33)] to determine the perturbations to the stiffness matrix due to the damage are shown in Fig. 3. The mesh plots are three-dimensional representations of the perturbation matrices. The rows and columns of the mesh plots correspond to the rows and columns of the perturbation matrices. The height of each peak represents the magnitude of the perturbation made to each matrix element. Note that with only two noise-free eigenvalues/eigenvectors, the algorithm is able to reproduce the exact damage. The algorithm demonstrates good performance when faced with noisy eigendata (lower plots). The percentage errors with respect to the exact stiffness damage for all studied cases are listed in Table 1. The perturbations to the damping matrix, ΔD_d, due to the damage are estimated by the extent algorithm [Eq. (31)] with exactly the same accuracy as those estimated for ΔK_d. In fact, the unscaled mesh plots of the computed ΔD_d's are the same as the ones shown in Fig. 3. This would be expected since, as reported earlier, the damage in the damping properties is proportional to the damage in the stiffness properties.

Table 1 Summary of percentage error with respect to the exact damage

Eigenvectors error	Percentage error with respect to exact stiffness (or damping): Upper longeron of bay three	Lower longeron of bay forty
0.0%	0.0	0.0
2.5%	6.6	7.42
5.0%	22.6	15.3

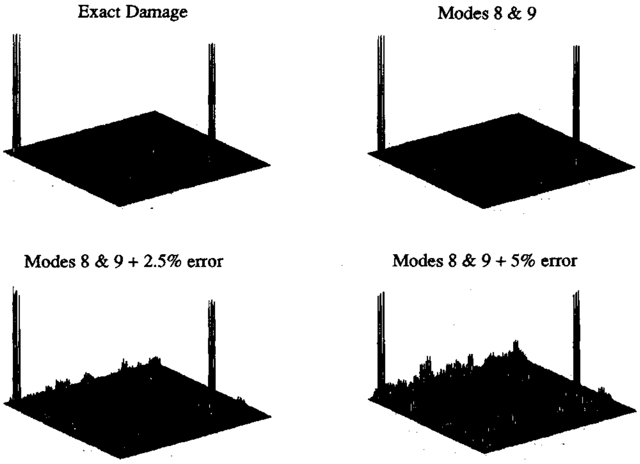

Fig. 3 Damage extent: modes 8 and 9.

B. Experimental Example: Mass Loaded Cantilevered Beam

This experiment was designed to illustrate the scenario of damage in mass properties. The structure used in this investigation is a cantilevered beam loaded with a nonstructural mass. A schematic of this beam is shown in Fig. 4. The dimensions and properties of the structure are summarized in Table 2.

An undamped FEM of the mass loaded beam was constructed using beam elements in conjunction with the properties of the nonstructural mass. The beam element has two DOFs at each node: bending and rotation. A 16-DOF undamped FEM was generated using the eight equal length element discretization shown in Fig. 2. The effect of the nonstructural mass was considered nonstiffening and concentrated at node 3. This mass was modeled by adding its mass and moment of inertia to the node 3 bending and rotation DOFs, respectively. The mass loaded beam, as described earlier, was considered the healthy configuration. Structural damage consisted of the removal of the nonstructural mass from the beam.

Experimental modal analysis of the beam was performed on the healthy and damaged beam configurations. Modal parameters were identified using frequency domain techniques and single-degree-of-freedom curve-fitting algorithms. The excitation source used was an impact hammer and the driving point measurement was at the free end of the beam. For each case, four modes of vibration were measured. Each mode consisted of a natural frequency and its corresponding mode shape with measurements at only the eight FEM lateral deflection degrees of freedom.

The number of measured eigenvector components (eight) is less than the number of DOFs in the FEM (16). In fact, only the bending DOFs of the beam were measured experimentally. Two approaches are available to correlate these dimensions: expansion of the measured eigenvectors[1,4-6] or reduction of the FEM.[7,8] It was found that a FEM reduction is better suited for this application. Thus, the FEM was reduced using the improved reduction system (IRS) method.[8]

It was found that the original reduced FEM does not accurately predict the dynamic behavior of the healthy beam. The first step in this study was to refine the original FEM. For the refinement process, it was assumed that the original mass matrix is an accurate representation of the structure's mass properties. The inaccuracy of the original FEM was believed to be due solely to modeling errors in the stiffness properties. The algorithm discussed in Sec. III.B.2. was used to correct the original reduced stiffness matrix. To get a symmetric refined stiffness matrix, the measured mode shapes (eigenvectors) were mass orthogonalized using the optimum weighted orthogonalization technique.[10]

Table 2 Loaded beam properties

Beam length—0.86 m
Beam mass/length—1.246 kg/m
Beam moment of inertia—1.458×10^{-9} m^4
Beam Young's modulus—69 GPa
Discrete mass weight—0.7938 kg
Discrete mass moment of inertia—1.1×10^{-3} kg-m^2

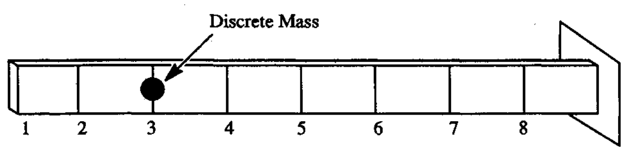

Fig. 4 Mass loaded cantilevered beam.

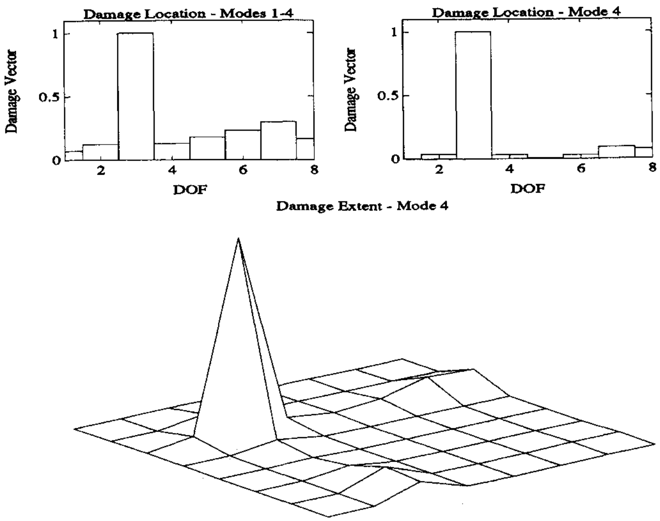

Fig. 5 Cantilevered beam: damage assessment.

The next step of this analysis is to determine the location of the structural damage. Using the refined FEM, the modal parameters of the four modes of vibration measured from the damaged beam were utilized to compute a cumulative damage location vector. Since the values of $\|z^j_{d_i}\|$, as defined in Eq. (2), are of different orders of magnitude the cumulative damage location vector should be computed using Eq. (3b). The upper-left corner of Fig. 5 displays the plot of the unit normalized cumulative damage location vector as calculated by Eq. (3b). From this plot, it is clear that DOF 3 has been affected by damage. This is exactly the bending DOF of the nonstructural mass. The small numerical elements at all other DOFs can be attributed to experimental measurement noise.

The final step of this analysis is to determine the extent of structural damage. With knowledge of the damage location, the rank of the true mass perturbation matrix, ΔM_d, is clearly one since there is only one DOF affected by the damage and that DOF is not connected to the cantilevered end. To compute a rank one ΔM_d only one mode of vibration is needed. As discussed earlier, the mode that should be used is the one that most cleanly demonstrates the damage detected by the cumulative damage location vector. Mode 4's $\boldsymbol{d}_i$ was determined to provide the best insight into the state of damage. The damage vector, $\boldsymbol{d}_i$, associated with mode 4 is shown in the upper-right corner of Fig. 5. The calculated ΔM_d using mode 4 data is shown in the lower plot of Fig. 5. It is clear that the extent calculation has concentrated the major changes at the DOF affected by the damage. From the extent calculation, the mass loss was estimated to be 0.7947 kg, which is within 1.1% of the actual mass loss.

V. Summary

In this work, the concept of the minimum rank perturbation theory (MRPT) is extended to determine the damage extent on the mass properties of undamped structures. The MRPT is also extended to address proportionally damped structures. For proportionally damped structures, it is assumed that the structural damage has affected two of the three property matrices. The process of computing the extent of structural damage using the MRPT is shown to be computationally attractive and hence suitable for large-order problems. The only computations required in determining the damage in a property matrix is an inversion of a matrix whose dimension is equal to the number of measured modes, along with matrix-matrix multiplications. The performance of the MRPT in assessing structural damage extent is demonstrated using both numerical and experimental data. The MRPT is also shown to be applicable for model refinement problems. The performance of the technique still needs to be evaluated on more complicated structures.

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